Finite Size Effects in Two Dimensional Turbulence

The term turbulence describes the state of spatio-temporal disorder characteristic of energetic fluid flows. Such flows occur in a wide variety of situations and play an essential role in many different physical and industrial problems. Atmospheric science, aeronautics, oceanography and astrophysics provide obvious examples. Turbulence differs from chaos in that chaotic systems are typically low dimensional whereas turbulence essentially involves the interaction of a large number of degrees of freedom. Conservation laws play an essential role in our understanding of the physics of turbulence. Since the Navier–Stokes equations governing fluid flows contain a dissipative viscous term, the notion of a conserved quantity requires explanation. In this context, a conserved quantity is something which is conserved by the nonlinear terms in the Navier–Stokes equation. Kinetic energy, $E = \frac{1}{2}|\vec{u}|^2$, is an example, where $\vec{u}(\vec{x},t)$ is the fluid velocity. For three dimensional turbulence it is the only important one for most applications.

From a statistical physics perspective, turbulence can be profitably thought of in terms of the transport of conserved densities between different scales of motion. The scales at which conserved quantities are dissipated are typically very different from the scales at which turbulence is forced. Viscous dissipation only becomes efficient at very small scales while friction is most efficient at large scales. Nonlinear interactions serve to transport conserved quantities from the source scale to the dissipation scale, a process referred to as a cascade. In three dimensional turbulence energy cascades from large scales to small where it is dissipated by viscosity. This allows the system to reach a stationary state having energy injection balanced by energy dissipation. There is no detailed balance. Rather there exists a range of scales, referred to as an inertial range, between the forcing scale and the dissipation scale through which a flux of energy flows.

The two-dimensional Navier–Stokes equations conserve additional densities, foremost among which is the enstrophy, $H = |\nabla \times \vec{u}|^2$. Additional conservation laws modify the physics considerably. Most importantly, in two dimensions, the nonlinear interactions act in such a way that energy is transferred from the forcing scale to larger scales where it is ultimately dissipated by external friction while the second conserved density, H, cascades to smaller scales where it is removed by viscosity. The transfer of energy to larger scales is called an inverse cascade while the enstrophy cascade to smaller scales is called a direct cascade. Assuming statistical isotropy of the turbulence, it is possible to make simple dimensional arguments for the energy spectrum, $E(k)$, of the stationary state where the fluxes of energy and enstrophy carried through the respective inertial ranges, usually denoted by ε and Q, are constant. For the direct cascade one finds $E(k) \sim Q^{\frac{2}{3}} k^{-3}$ and for the inverse cascade, $E(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$. For this research we are primarily interested in the inverse cascade. By transporting energy from incoherent small scale forcing to larger scales, inverse cascades facilitate the formation of large scale coherent structures, in this case vortices. The generation of large vortices in two dimensional turbulence whose lifetime is often many orders of magnitude greater than the coherence time of the forcing is good example of the emergence of “order from chaos”.

In most studies of the two–dimensional inverse cascade, the scale of the largest vortices in the system is limited by the external friction. As a vortex grows larger, the drag on the fluid layer increases until eventually it is sufficient to balance the energy flux carried by the inverse cascade. At this point the vortex cannot grow any further. If, however, the external drag is decreased sufficiently or removed entirely (something which can be easily done in numerical simulations) then the vortices continue to grow until eventually the inverse cascade reaches the largest scale of the system. This is the box size in a numerical simulation or the size of the container if such a scenario could be realised in a laboratory. In this situation, the inverse cascade cannot proceed any further resulting in what we call a blocked inverse cascade. We have performed a series of numerical experiments to investigate inverse cascades blocked by finite size effects. The main conclusion of our work so far is that finite size effects can provide a mechanism for strongly enhancing the stability and coherence of the large scale structures present in two-dimensional turbulence. In fact turbulence can be entirely suppressed at the largest scales resulting in a large scale flow which is smooth and effectively deterministic.

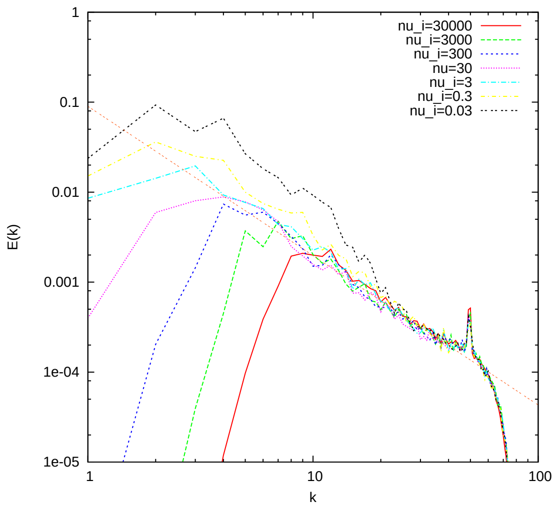

Figure 1: $E(k)$ for different dissipation strengths.

Figure 1 shows the energy spectrum, $E(k)$, for various values of the large scale dissipation. For strong dissipation, it scales like $k^{-\frac{5}{3}}$ as expected from standard theory. As dissipation is decreased, the inverse cascade reaches the size of the system and is blocked. The spectrum then crosses over to k^{-3}, characteristic of a smooth large scale flow. The corresponding vorticity field is shown in figure 2. It clearly shows how the blocked inverse cascade organises itself to produce a very intense vortex dipole.

Once the large scale vortex dipole emerges it is very stable. If the dissipation is entirely absent then the amplitude of the vortices constituting the pair continues to grow indefinitely at a rate proportional to $\sqrt{t}$. This is as one would expect giventhat the total enstropy of the system grows linearly in time (constant injection). If there is some small amount of dissipation present, then the growth eventually saturates. Small scale fluctuations produced by the forcing remain in the system. However as the amplitude of the large scale flow grows, it comes to completely dominate the fluctuating part so that at large scales, the flow no longer appears turbulent. In this sense, blocked inverse cascades tend to “suppress” turbulence. Nevertheless, the effect of the small scale fluctuations is felt at large scales through the fluctuations of the relative positions and velocities of the two vortices in the pair. The separation of the two vortices remains around half the scale of the box but fluctuates significantly. Furthermore, the pair exhibit a uniform mutual translation whose direction and magnitude also fluctuate.

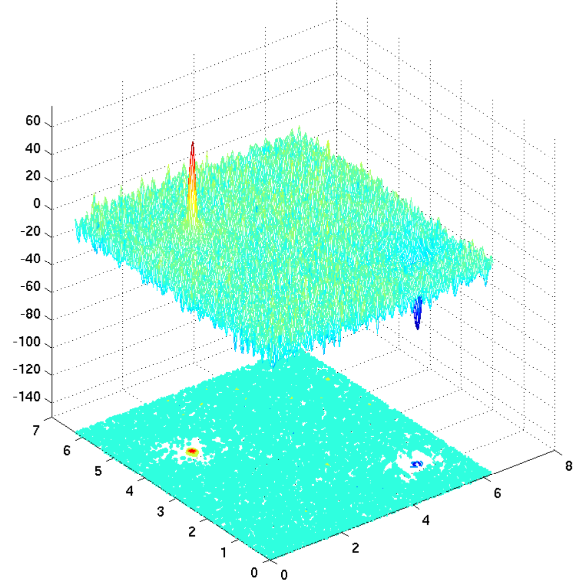

Figure 2: Vortex dipole resulting from a blocked inverse cascade.

From the numerical measurements it is clear that the structure of the vortices is such that the vorticity seems to decrease as a power law as one moves away from the centre of the vortex. The exponent of the decay is approximately 1.25, a number which currently lacks a convincing theoretical explanation. This $r^{-1.25}$ radial vorticity profile is very robust. It persists in time as the vortex dipole grows in amplitude. Furthermore it remains if the driving is turned off allowing all small scale fluctuations to decay.

It is an interesting question to ask how universal is the behaviour described here. It is important to remember that all the physics of blocked inverse cascades and the corresponding coherent large scale flow comes from the finite size effect and presumably depends on the boundary conditions which constrain the system. The vortices which emerge from non-slip boundary conditions in a box will probably differ in detail from those emerging from the periodic boundaries used in our simulations to date. Even so, we expect that the phenomenon of finite size effects leading to the generation of coherent, highly stable large scale structures is generic.

Contact Information: Colm Connaughton – CNLS and T13, MS-B213, Los Alamos National Laboratoy, Los Alamos, NM, 87545, Phone: (505) 667-7075, E-mail: colm@cnls.lanl.gov.